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Slicing a Puzzle and Finding the Hidden Pieces

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SLICING A PUZZLE AND FINDING THE HIDDEN PIECES

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ABSTRACT

The research conducted was to investigate the potential connections between group theory and a puzzle set up by color cubes. The goal of the research was to investigate different sized puzzles and discover any relationships between solutions of the same sized puzzles. In this research, first, there was an extensive look into the background of Abstract Algebra and group theory, which is briefly covered in the introduction. Then, each puzzle of various sizes was explored to find all possible color combinations of the solutions. Specifically, the 2x2x2, 3x3x3, and 4x4x4 puzzles were examined to find that the 2x2x2 has 24 different color combination possibilities, the 3x3x3 puzzle has 11,612,160 color combinations, and the 4x4x4 has at least 1,339,058,552,832,000 color combinations. We cannot say exactly how many the 4x4x4 puzzle will have due to the insufficient certainty of the possible solutions of the 4x4x4 cube.

After inspecting each solution for the cube, it was found that the 2x2x2 puzzle had 4 transformations (or elements, in group theory terms), and the 3x3x3 puzzle had either 9 or 27 elements. The number of elements for the 3x3x3 puzzle was dependent on its original set up. If not every cube moved in the same direction horizontally and vertically, the puzzle would have 27 elements. Since the research was not sufficient enough to find a definite number of set ups that the 4x4x4 cube could have, there was not enough information to build upon to find a collection of the elements or groups that this puzzle would be isomorphic to. However, the other two puzzles, the 2x2x2 and 3x3x3, were successfully mapped to another group, proving that these groups are isomorphic. The 2x2x2 puzzle mapped to the group $Z_2 \oplus Z_2$. The 3x3x3 puzzle is mapped to either

the group $Z_3 \oplus Z_3$ or $Z_3 \oplus Z_3 \oplus Z_3$, depending on which group the original set up belonged to.

KEYWORDS: Combinatorics, Group Theory, Recreational Mathematics

INTRODUCTION

A popular question always being asked is how math is applicable to real life situations, especially branches of math such as Abstract Algebra. With such a complex subject, it's hard to imagine that it could be related to something as simple as a puzzle. However, through this color cube puzzle presented in this article, we can see the complex concepts of Abstract Algebra being used to help find its solutions. This puzzle that is going to be investigated begins by setting up a 2x2x2 cube, made up of 8 cubettes of 4 different colors. Throughout this article, we will refer to cubettes as the smaller cubes that are arranged to create the cube puzzle. The goal is that when the cube is set up, each color is represented exactly once on each of the six faces of the cube. Once the 2x2x2 has been solved, the 3x3x3 can be attempted. The same restrictions apply with each color needing to be represented on each face exactly once, except now there are 9 different colors and 27 cubettes total. The goal of this research is to take a deeper look into this puzzle and examine the connections to different symmetries and mathematical groups (Brown & Hathaway, 2012).

In order to see how group theory relates to this puzzle, it is important first to understand what properties a group consists of. A group is a set of elements paired with an operation, where the pair meets 4 specific requirements. The 4 properties of a group are (1) the group must contain an identity, (2) every element must have an inverse, (3) every element must be associative, and (4) the group must have closure (Gallian, 2010). Operations, which we will call transformations, on a solved puzzle lead to a group. Before relating group theory to the color cube puzzle, solutions to the puzzle should be further explored. The solution that one begins with will affect what elements the cube's group contains. Since we find that different solutions of the puzzle lead to different groups, it is important to find a systematic way of looking at these different solutions.

REVIEW OF LITERATURE

The idea for this puzzle, as described in the introduction, was presented to me through an article written by Dr. Justin Brown and Dr. Dale Hathaway of Olivet Nazarene University. This puzzle brings Abstract Algebraic concepts into a tangible, manipulative form, creating a relatable way for students to investigate Group Theory, as well as symmetry. The term *group* has changed over after first being referred to by Galois in 1830. Gallian defines a group to be a set together with a binary operation that is associative, such that there is an identity, every element has an inverse, and any pair of elements can be combined without going outside the set (Gallian, 2010). When a solved cube is generated, operations performed on the solved cubes lead to the family of groups $Z_n \oplus Z_n$, where n is the length of cubettes on each side of the cube (Brown & Hathaway, 2012).

Symmetry has a large part in connecting this puzzle to Abstract Algebra. Symmetry is typically only thought of when looking at images or patterns that repeat. However, symmetry goes much beyond this concept. Take for example, the 3-permutahedron, which is shown as a regular hexagon with its vertices labeled as permutations of {1, 2, 3} (Crisman, 2011). This symmetry group is D_{12} with the normal subgroup S_3. This 3-permutahedron can also use left- and right-multiplication with direct products, where each product is distinct. These symmetry groups can be represented in Cayley Tables, or Cayley Graphs, as they are occasionally referred to. These tables are "a combinatorial graph or digraph representing the action of multiplication of the elements of a given group G by elements of a generating set S for G," (Conder, 2007). These tables generate a visual to compare symmetry groups that are isomorphic to one another. The elements of a symmetry group are listed on the top and side of these tables with their products displayed in the corresponding cells of the table. Each of these products will be an element already listed in the group in order for the set of elements to be a group.

Puzzles are typically solved using some logic, reasoning, or trial and error. However, puzzles can often be solved using math, as well. A common example of a puzzle that can be solved using either strategy is the Rubik's cube. This puzzle has been a popular game that has been solved by many using trial and error, but has also been studied in its complexities by many mathematicians. Permutations are one of the ways of looking at all the possible moves that can be performed on a puzzle in order to solve it. In the case of the Rubik's cube, any twist is a move, meaning any combination of twists is a permutation of the puzzle (Davis, 1982). After figuring out the number of permutations and properties of the permutations, they can be used to determine solutions to the puzzle.

RESULTS

SOLUTIONS

In each puzzle, no matter the size, there are different ways of looking at generating solutions for the puzzle. For the sake of consistency, we will refer to each layer being a translation of the layers preceding it. A translation (in this context) refers to in which directions (horizontally and vertically) the cubettes are shifted. The size of the cube determines what translations produce solutions to the puzzle. In all cases, we know that each cubette needs to be moved in either the same direction vertically or the same direction horizontally, but not necessarily in the same direction both horizontally *and* vertically, although that produces a unique solution as well.

Also, another important note about the cubettes' translations is that they relate to modular addition. This may be easier understood in pictures than in words alone, so let's take a look at the 2x2x2 example in the next section to understand.

2x2x2

As we look at a 2x2x2 case, it is quickly made obvious that there is only one possible solution for the second layer once the first layer of the four different colored cubettes has been decided. Let's look at the different possible translations to prove that this is a true statement in Figure 1.

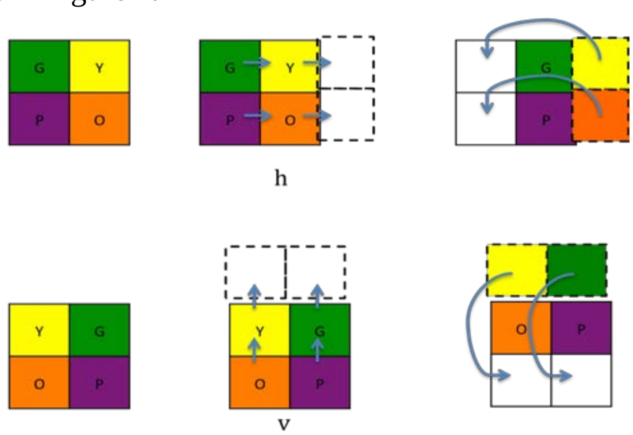

Figure 1

Above shows the translation (h,v). After looking at each of these translations of (h,v), (h',v), (h,v') and (h',v'), we notice that each translation ended up giving us the same second layer. This tells us that in the 2x2x2 case, (h,v) = (h',v) = (h,v') = (h',v').

3x3x3

Now, as we look into the case of the 3x3x3 cube, we see that the same is not true of the translations. Since this cube has 3 layers, there is much more room for different set ups for solutions. Just as with the 2x2x2, we can look at the different translations of (h,v) and all of its opposites for the cubettes. Now that there is a third layer to each side of the cube being added, we have to consider translations such as (2h, 2v), as now we can move a cubette in the same direction twice and it will be in a new location. However, as we consider each of these new translations, we can see that they are each equivalent to one of the original translations of (h,v), (h',v), (h,v'), or (h',v'), as follows:

2h = h' 2v = v' 2h' = h 2v' = v

Therefore, we can say that each translation can be put in terms of h, h', v, and v'. There also exists a translation for the cubettes such that on any given diagonal (same diagonal for each layer), the cubettes on the diagonal will follow one translation, while every other cubette not on the diagonal will follow the exact opposite translation. For example, if each cubette on the diagonal follows the translation (h,v), every other cubette will follow the translation (h',v').

As previously stated, each of these new translations will lead to a different set up for a different solution. Another way of stating this is given a particular first layer in the 3x3x3, the second and third layers can be decided by following any of these translations. Notice that once the second layer has been set up, the third layer has been decided. Later on we will discuss how each cube relates to a group and allows for specific transformations on the cube (different from the translations on the cubettes), which is all dependent on the initial set up and transformations on the cubettes.

4x4x4

In the case of the 4x4x4 cube, we run into a different situation than the 3x3x3 cube or the 2x2x2 cube because in an *n* x *n* x *n* cube, when *n* is even, a translation of 2h or 2v (of some form) will cause a layer to be duplicated, violating the rules of a valid solution. Therefore, we now must consider the translation of 3h or 3v. With this translation done to the first layer, the second layer is created and is unique to the first. As this is repeated throughout the layers, each layer remains unique and gives us a solution that follows the rules of a valid cube. Just as we could in the 3x3x3 cube, we can still write each of these 3h or 3v moves in terms of simply h and v, as follows:

3h = h’ 3v = v’ 3h’ = h 3v’ = v

Therefore, there is a way to write each translation for the cubettes in the 4x4x4 case in terms of h, v, h’ and v’. This helps us see that there are less possible translations than one might originally think, as well as simplifies the notation.

COLOR COMBINATION POSSIBILITIES

After dissecting the possible translations that can be performed on the cubettes in order to form a cube that is a valid solution, we can now analyze the exact number of color combinations there can be for solving each puzzle. Obviously, as the puzzle gets larger, so do the number of color combination possibilities. Let’s begin by looking at the 2x2x2 case.

2x2x2

In this color cube, there are two layers. Since we build the second layer based on the first layer, we must look at the number of color combinations that can result from just one layer. In this primary layer, there are four spots and 4 colors to choose from, one color for each spot. This calls for the use of a permutation, since the order does not matter in this layer. When calculated, we find that ${}_4P_4 = 4! = 4 \cdot 3 \cdot 2 \cdot 1 = 24$. This means that given 4 different colors to solve a 2x2x2 cube, there are 24 different ways to arrange the cubettes in the first layer.

Now, since the second layer is dependent on the first, we can move on to finding the number of possible color combinations of the cubettes in the second layer. As stated in the solutions section for the 2x2x2, we know that once the first layer of the 2x2x2 has been set, the second layer has only one possible way of being arranged such that the cube is a valid solution. Therefore, we take the number of possibilities for the first layer, 24, and multiply it by the number of possibilities for the second and final layer, which is 1, giving us a result of 24 different possible color combinations in the 2x2x2 puzzle.

3x3x3

As we look into the 3x3x3 cube's color combinations, we can quickly see that there are many more than 24 color combinations. First off, we have introduced 5 more colors since each layer has a total of 9 different colors. Since there are 9 spots to be filled, each with one color and order does not matter, the number of different color combination possibilities is ${}_9P_9 = 9! = 9 \cdot 8 \cdot 7 \cdot 6 \cdot 5 \cdot 4 \cdot 3 \cdot 2 \cdot 1 = 362{,}880$. Now that we've taken into account all possibilities for the primary layer, we consider the 2nd level of the cube. Since we know that, in order to create a solution, the 2nd layer must be a specific transformation of the first layer.

As we found previously, these translations for the 3x3x3 cube have to be written in the form (h,v). There are the cases where all the cubes follow the same translation, where they both move in the same directions horizontally and vertically. This gives us 4 different translations. Secondly, there are 12 cases where the translations can move in the same direction horizontally and 12 separate cases where they move in the same directions vertically.

Finally, there is the last translation for the cubes called the diagonal translation. In this translation, the 3 cubettes on one of the 2 diagonals all follow the same translation, say (h,v), while the rest of the cubettes not on the given diagonal follow the exact opposite translation, (h',v'). Since there are 2 diagonals and each diagonal can move either (h,v) or (h',v'), we have 4 different diagonal translations. Putting all these different translation combinations together, we have 32 different possible set-ups for the 2nd layer. The 3rd layer has already been decided once the 1st and 2nd layers have been set because each color now has a specific spot that it needs to be in order to make sure the cube meets the puzzle's requirements of having each color represented on each side. Therefore, the total number of color combination possibilities for the 3x3x3 cube would be $9! \cdot 32 \cdot 1 = 11{,}612{,}160$.

4x4x4

We can approach the 4x4x4 case in the same way we did the 3x3x3. First we evaluate the number of color combinations that can make up the first layer. This 4x4x4 cube has 16 different colors for the cubettes in the puzzle, meaning we have 16! = 20,922,789,888,000 different color combinations for the first layer alone!

Now we look at the possibilities for the 2nd layer as we did before by evaluating the number of different translations that can be performed and produce a solvable puzzle. In this 4x4x4 cube, we get 4 translations again where the transformation moves each cubette in the same direction horizontally and vertically. Now each translation has 16 different combinations of translations where they move in the same direction vertically, but one row shifts in a different direction horizontally, as well as 16 more for translations in the same direction horizontally with one row moving in a different direction than the rest vertically. There are 12 more possibilities for the translations where the cubettes all move in the same direction vertically, but half move in one direction horizontally and the other half move the opposite direction horizontally, as well as another 12 for the cubettes moving in the same direction horizontally and half move one way vertically, while the other half moves the opposite way vertically. As far as can be examined for the purpose of this research, these are the only translations that I have found to lead to a solution in the 4x4x4 cube, but this has not been an exhaustive search for solutions for this puzzle, so we cannot say this is the definitive number of translations for this puzzle.

Finally, we consider the moves that can take place in the diagonal translation. While there are more possible translations (such as (2h,2v) or (h,3v)) that cause different set-ups, these translations cause overlaps in the cubettes or make set-ups that do not create a solvable puzzle. Therefore, there are still only 4 diagonal translations in the 4x4x4 case again, where each of the two diagonals can be either (h,v) or (h’,v’), producing four different translations. This gives us the total of 62 different translations that can be performed on the first layer to generate the rest of the cube.

Since each layer needs to be produced using the same translation as all the other layers, this tells us that after the first and second layer have been decided by first choosing a color set up for the primary level and second choosing a translation to manipulate each layer, the rest of the remaining two layers have also been decided. Therefore, the total minimum number of color combination possibilities that can be created in a 4x4x4 solvable cube is $16! \cdot 62 = 1{,}339{,}058{,}552{,}832{,}000$.

TRANSFORMATIONS

Now that we've established an in depth look at the set up possibilities for each puzzle, we can finally begin to look into the movements that can be performed on the cube that still yield a solution. We call these movements that are performed on the cube transformations of the cube, and as the cube's dimensions get bigger, the more transformations are possible to perform while keeping the integrity of the puzzle. For example, in any solvable cube, the front face of cubettes should be able to be moved to the back of the cube, and still give us a solution to the puzzle. Later on, we will discuss how each of the cubes relates to a specific group, depending on the set up, and how these transformations affect the related group. As usual, let's begin by looking at the 2x2x2 cube in order to see these transformations in their simplest form.

2x2x2

In the 2x2x2 puzzle, we first begin by defining that a transformation needs to be the movement of a layer or "slice" from one side of the cube to another. By limiting our transformations to this restriction, we will be able to get a better look at the similarities between the color combinations we obtain. We should also notice that there is only one case to be studied in the 2x2x2 puzzle since there is only one possible general set up for the 2x2x2 solution (once given a particular base).

Now we can begin by looking at the transformation mentioned previously, where we take the front “slice” of cubettes, meaning the four cubettes on the front face of the cube, and moving them to the opposite side of the cube. Since this transformation gave our original 2x2x2 cube a new color arrangement that still left the puzzle solvable, we now have a valid transformation for this puzzle. We can call this transformation where we move the front face of cubettes to the back “F”. Now notice that by performing “F” again on the 2x2x2 gets us back to the original color set up we started our cube with. We will call this original set up the “identity”. This is because once we performed “F” once, the front slice was in the back, and the back slice was in the front. Performing the same operation again, we get the front slice shifted back to the front again with the back slice in the back, hence we have our original cube.

Now that we’ve established “F”, we want to look for another transformation that can be performed that will give us a new color combination than one of the ones already found by our identity and by the transformation of “F”. If we move the left slice of the puzzle to the right side (and in turn shifting the right side to the left), we have a new color arrangement for our 2x2x2, meaning we have a new transformation. Let’s call this transformation “L”, where the left slice is moved to the right side of the cube.

Now that we’ve established the transformations F and L, we can combine these transformations to see that this leads to another combination. Since this set up is a combination of the two transformations, which we will call F*L, we perform the first transformation on the cube, and then follow that with the second. For every new color combination we get from a given transformation, we can call each of them an element. This means that we are defining an element to be a color combination that results from a transformation on a solved puzzle. That means a cube must have the same 4 colors as its original set up in order to be an element of that cube.

Now that we have determined all the elements we can get from F and L (3 so far, which are F, L, and F*L), we want to determine if there are any other transformations that will lead to a new element of the 2x2x2. The next movement we naturally want to claim to be a transformation would be moving the top slice of the puzzle to the bottom. While this does create a set up that keeps the puzzle in a solvable set up, it creates an element that we've already discovered. If we look at the positions of the cubettes in this set up from the top to bottom movement, we find it to be identical to the positioning of the cubettes of the F*L element. Since a transformation must create a unique element, the top to bottom movement is not a transformation in the 2x2x2 puzzle.

In order to find all the possible transformations for this cube, we need to examine every set of four cubettes that make up a face of the cube. While our transformations that we've already discussed may seem that we have only examined the left and front faces of the cube, when in reality, when we look at performing a transformation where we move the right slice to the left, or the back face to the front, we once again get elements that have already been produced. In the case of the 2x2x2, we can see that the transformation L yields the same element of R because in both cases, the right slice and the left slice switch places. The same situation takes place with F and R (where R is the rear face) as the front and rear slices switch places in both transformations. Also, just as before, since the movement of the top layer moving to the bottom did not result in a new element, the movement of the bottom layer moving to the top will also not result in a new element. Now we can see that we have established the 2 transformations in the case of the 2x2x2 cube.

3x3x3

Now that we've set up the groundwork for transformations in the 2x2x2, we can dive into transformations in the 3x3x3. We should notice right away that these transformations will be more complex than the 2x2x2 transformations since there is another layer added in. In the 2x2x2 with the transformation L, each layer just swapped places. In the 3x3x3, when the left layer moves to the right side, now the middle layer becomes the left layer, and the right becomes the middle. However, this does not mean they aren't a solution, it solely means it needs further exploration.

I have found that, similarly to the 2x2x2, the movements L and F lead to new elements for the cube. As a reminder, these new elements are manipulations of the identity that create different combinations of the same color cubettes that the identity held, as well as still holding a solution to the puzzle. Also, the combination of these 2 transformations together creates another element. However, unlike before, if we operate the same transformation twice, we do not come back to the original element again, as we did in the 2x2x2. If we operate the transformation L with itself, L^2, we get a completely new element for this cube. The same is true for the F transformation. However, once we reach L^3 (or F^3), now we come back to the original set up, meaning we do not have a new element.

Now that we have these elements, we can now test to see if, just as before in the 2x2x2, combination of transformations operated with each other will also lead to new elements. This gives us new elements such as F*L, F^2*L, F^2*L^2, etc. Another way to view all of these elements for this group is in a Cayley Table (Conder, 2007), seen here in Table 1.

Table 1 – Cayley Table for a 3x3x3

*	e	L	L^2	F	F^2	L*F	L^2*F^2	L^2*F	L*F^2
e	e	L	L^2	F	F^2	L*F	L^2*F^2	L^2*F	L*F^2
L	L	L^2	e	L*F	L*F^2	L^2*F	F^2	F	L^2*F^2
L^2	L^2	e	L	L^2*F	L^2*F^2	F	L*F^2	L*F	F^2
F	F	L*F	L^2*F	F^2	e	L*F^2	L^2	L^2*F^2	L
F^2	F^2	L*F^2	L^2*F^2	e	F	L	L^2*F	L^2	L*F
L*F	L*F	L^2*F	F	L*F^2	L	L^2*F^2	e	F^2	L^2
L^2*F^2	L^2*F^2	F^2	L*F^2	L^2	L^2*F	e	L*F	L	F
L^2*F	L^2*F	F	L*F	L^2*F^2	L^2	F^2	L	L*F^2	e
L*F^2	L*F^2	L^2*F^2	F^2	L	L*F	L^2	F	e	L^2*F

This is the Cayley Table of one example of a solved 3x3x3 cube, since the Cayley Table will differ if other setups are used. The cube this Cayley table is based on was created by one of the four translations where all the cubettes move in the same direction horizontally and vertically. As we can see, this cube has 9 distinct elements, meaning there are 9 transformations that give this cube a new color combination without breaking the properties of the puzzle. The Cayley Table shows the different elements operated with each other and what they would yield as a result.

Now, in order to make sure the group is complete, we need to see that there are no more possible elements, meaning there are no other transformations that we can perform on the cube to get a new color combination in the cube. The only transformation we have not considered yet is the move of the top slice to the bottom, which we call B. In this specific case, where the cube was set up with the transformation of each cubette moving in the same direction horizontally and vertically, we find that B actually reveals a color combination of a cube we've already seen. However, this is not always the case when we look at the different setups we can have in the 3x3x3.

While looking at the setups for the cubes where the translations were either all in the same direction vertically and not horizontally, all in the same direction horizontally and not vertically, and where the translation was on the diagonal, we will see that L and F (and every product of L and F) still create new elements, just as they had in the cubes with the translation of every cubette moving in all the same direction horizontally and vertically. However, once we look at the transformation B, we find our results to be different than before. Now, when we move that top slice to the bottom of the cube, we do in fact get a new color combination than any of the previous elements have given us. This results in B being a new element.

Just as how L and F had different combinations of those transformations to give us more elements, we now must incorporate B into each of those elements to find all the possible new elements this group can give us. Now instead of 9 elements, we have 27 elements. This shows us that there is a significant difference in the 4 setups where the translations move the cubettes in the same direction horizontally and vertically compared to the other 28 setups. In order to further explain these differences, we can look into the connections that have been made between this puzzle and Group Theory.

CONNECTIONS TO GROUP THEORY

In Group Theory, a group is not simply a collection of items, but rather is much more complex than that. Here, we are considering a group to be a set of elements paired with a binary operation. As we touched on earlier, in order for a set to be considered a group, it must keep the four properties of a group true. These 4 properties of a group are (1) the group must contain an identity, (2) every element must have an inverse, (3) every element must be associative, and (4) the group must have closure (Gallian, 2010).

The first requirement states the set of elements must include an identity element. This identity is an element that, when operated with any other element of the group, the resulting product is the original element (i.e. if $a, e \subset G$, meaning a and e are elements in G where G is the group, a is any element in G, and e is the identity element, then $a * e = a$, $e * a = a$).

Secondly, each element in the group must have an inverse. This property states that for every element in the group, there is an inverse element that it can be operated with to produce the identity element (i.e. if $a, b, e \subset G$, b is the inverse of a, and e is the identity element, then

$a * b = e$, $b * a = e$).

The third requirement states that the elements under the given operation are associative (i.e. if $a, b, c \subset G$, $a * (b * c) = (a * b) * c$). Finally, the last requirement for a set of elements to be considered a group states that the group must have closure. In order for a group to be closed, when any two elements of the group are operated together, they must produce another element in that same group. If the set of elements meets each of the preceding criteria, it can be named as a group in Group Theory terms.

To put the term "group" into a practical example, we can look at the example of the dihedral group, known as D_4, which contains all the elements of possible rotations and flips of a square. The elements contained in D_4 are as follows: R_0, R_{90}, R_{180}, R_{270}, H, V, D, and D'. The R_x elements describe the square being rotated (in a counter-clockwise motion) by each degree noted as x. The R_0 element rotates the square 0°, leaving the square in its original position. This makes R_0 the identity element of this group. If R_0 is the identity element, then we can operate it with any other element of the group, and the product will be that second element chosen from the group. For example, given the elements in D_4 where R_0 is the identity as described and a is any other element of the group, $R_0 * a = a$.

H, V, D, and D' are different flips of the square over a specific line of symmetry. H refers to a horizontal flip, V a vertical flip, D a flip across the diagonal, and D' a flip across the other diagonal. R_0, R_{90}, R_{180}, and R_{270}, are all different rotations, where x in R_x refers to the degree of which the square is rotated. Figure 2 demonstrates these elements of D_4.

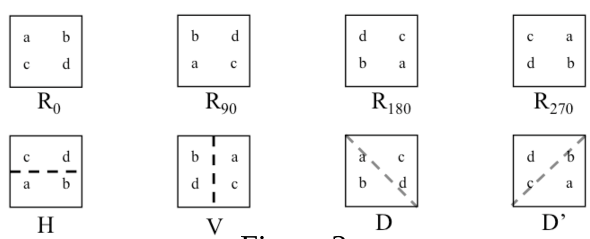

Figure 2

Now, we look to see if each element has an inverse. We can see that each flip is its own inverse, as well as the identity element and R_{180}. This leaves us with R_{90} and

R_{270}. We know these two elements are inverses of each other because when we operate them together ($R_{90} * R_{270}$ or $R_{270} * R_{90}$), meaning we rotate the square 90° and 270°, a total of 360°, we arrive back at R_0, the identity element. Therefore, each element does, in fact, have an inverse.

Next, we can state that elements of a group are associative. We know that these elements are each associative because throughout all of the elements, the product is not affected if one element is operated before the other elements. For example, we can look at V * H and H * V. We can see below in Figure 3 that both products yield the same end result.

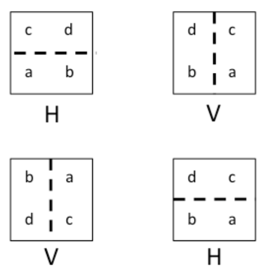

Figure 3

Finally, we know that this group is closed because we have every possible combination of every letter being in every corner listed, so therefore when any two elements are operated on together, they must create another element in the group. Thus, with all requirements met, D_4 is a group. (Gallian, p. 30)

Now that we understand the fundamental parts of being a group, we can describe how the solutions to this color cube puzzle is, in fact, a group. Again, the four requirements for being a group are having an identity, being associative, having an inverse for each element, and being closed. First, we can find the identity element by looking at which element, when performed with another element, does not change the cube. In our color cube groups, for the 2x2x2 and each different set up in the 3x3x3, the identity element is “e”, the transformation that does not move any of the layers of cubes. We know this is the identity element because we can see that when operated with any other transformation, the result is the other transformation the identity operated with.

Next, we will investigate if these groups are associative. By definition of associative, any two elements of the group should be able to be operated together in any order and the end result will be the same. We know that each of these groups is associative because no matter the order that the transformations are operated in, the final set up is the same. For example, if we refer back to a Cayley Table, as we did in Table 1, for the transformations of a 3x3x3 cube with a particular set up, we can see that order of transformations does not affect the final set up of a color cube.

Table 2 – Cayley Table for a 3x3x3

*	e	L	L^2	F	F^2	L*F	L^2*F^2	L^2*F	L*F^2
e	e	L	L^2	F	F^2	L*F	L^2*F^2	L^2*F	L*F^2
L	L	L^2	e	L*F	L*F^2	L^2*F	F^2	F	L^2*F^2
L^2	L^2	e	L	L^2*F	L^2*F^2	F	L*F^2	L*F	F^2
F	F	L*F	L^2*F	F^2	e	L*F^2	L^2	L^2*F^2	L
F^2	F^2	L*F^2	L^2*F^2	e	F	L	L^2*F	L^2	L*F
L*F	L*F	L^2*F	F	L*F^2	L	L^2*F^2	e	F^2	L^2
L^2*F^2	L^2*F^2	F^2	L*F^2	L^2	L^2*F	e	L*F	L	F
L^2*F	L^2*F	F	L*F	L^2*F^2	L^2	F^2	L	L*F^2	e
L*F^2	L*F^2	L^2*F^2	F^2	L	L*F	L^2	F	e	L^2*F

We can look at many different combinations in this table, 81 combinations to be exact, and we will see that every time a pair of transformations is operated on in either order, the final set up is the same. For example, if we look at the left column and choose the transformation L*F, and operate it then with the top row element L^2*F, we get the element F^2. Now, if we were to switch the order of those transformations and operate L^2*F with then L*F, we still get the element F^2. We can prove this with every transformation in the Cayley table above, as well as a Cayley table created for the other set ups for the 3x3x3, as well as the 2x2x2 puzzle. The 4x4x4 puzzle would follow the same rules, for the set ups that have been figured out for the 4x4x4. As we saw with the differences in the 2x2x2 and the 3x3x3 cube, the increased number to the dimensions of the puzzle creates a massive amount of differences in the puzzles, as well as a whole new level of complexities to be explored. For the sake of brevity in this article, the 2x2x2 and 3x3x3 puzzles will remain the focus of discussion.

Now that we've discussed the identity and associative properties, we can look at the inverse property for this group. The inverse of an element should be able to "undo" that particular element, meaning when one element is operated, the inverse of that element will return the cube to its set up prior to being operated on. An element and its inverse when operated together will result in the identity element. So, in looking at our Cayley Table, we should be able to confirm these results by finding an identity element in every column and every row, meaning that each element has another element it can be operated with to produce the identity element. Since our Cayley table does prove this in figure, we can say our group has the inverse property.

Finally, in order to prove that the solutions to our puzzles are groups, we need to show that they are closed. If a group is closed, each element, when operated with any other element in the group, will produce an element already found in the group. This means that if we operate any two elements together in our left column or top row, we should find that their product is already an element included in the left column or top row. Our Cayley table confirms this result for the particular set up of a 3x3x3 cube, as all the products in the table are elements of the group, so we can conclude this group would have the closed property. These same findings resulted for the 2x2x2 cube, as well as the other set ups for the 3x3x3 cubes.

Since each of these groups of elements has each property of a group, we can confirm that they are what Abstract Algebra classifies a group to be. The interesting relationship that I found for each of these groups was the groups that each puzzle was isomorphic to. Groups are isomorphic to one another if they share the same number of elements, where each element has a corresponding element with the same order. Each element has a certain order, meaning the number of times an element has to operate with itself to produce the identity element.

After researching groups with different elements and orders, it can be shown that the group $Z_2 \oplus Z_2$ is isomorphic to the 2x2x2 color cube puzzle. In both groups, there are four elements, each with an order of two (besides the identity element). This color cube puzzle is also isomorphic to the Klein group, which has 4 elements, each with an order of two as well. The following Cayley tables show the similarities in the groups and show their isomorphism.

Table 3 - Klein Group

*	1	a	b	ab
1	1	a	b	ab
a	a	1	ab	b
b	b	ab	1	a
ab	ab	b	a	1

Table 4 - 2x2x2 Color Cube Group

*	e	L	F	B
e	e	L	F	B
L	L	e	B	F
F	F	B	e	L
B	B	F	L	e

Table 5 - $Z_2 \oplus Z_2$

*	(0,0)	(0,1)	(1,0)	(1,1)
(0,0)	(0,0)	(0,1)	(1,0)	(1,1)
(0,1)	(0,1)	(0,0)	(1,1)	(1,0)
(1,0)	(1,0)	(1,1)	(0,0)	(0,1)
(1,1)	(1,1)	(1,0)	(0,1)	(0,0)

In each of these groups, we can see which elements would map to one another. We find that each of the following elements is a mapping of the other:

1 → e → (0,0)

a → L → (0,1)

b → F → (1,0)

ab → B → (1,1)

Since we can find a mapping for each element to another in each group, these three groups can be called isomorphic. (For a definition of isomorphism, see page 123 of (Gallian, 2010).) We can find similar findings in our first set up for the 3x3x3 cube, where each cube on one layer is translated to the next layer in the same pattern for each row. Looking at this specific set up, we found it had nine elements: e, L, L^2, F, F^2, L*F, L*F^2, L^2*F, and L^2*F^2. Another group with the same number of elements and the same corresponding orders was the group $Z_3 \oplus Z_3$. In this group, the nine elements are as follows: (0, 0), (0, 1), (0, 2), (1, 0), (1, 1), (1, 2), (2, 0), (2, 1), and (2, 2). We find that each of the following elements maps to one another:

e → (0,0)	F → (1,0)	$L*F^2$ → (1,2)
L → (0,1)	F^2 → (2,0)	L^2*F → (2,1)
L^2 → (0,2)	L*F → (1,1)	L^2*F^2 → (2,2)

The interesting findings come into play when we take a look at the differences in the groups that are isomorphic to the groups with a different set up for the 3x3x3 cube. Instead of setting up each puzzle by translating each cube into its new row by performing the same translation, this solution is set up by translating each cube all the same direction horizontally or vertically, and moving one row the opposite direction vertically or horizontally. We found that there are 12 different set ups in which we can set up the solution in this manner. We also found four more solutions where we could set up the cube by doing a translation on the diagonal, as previously mentioned in the section about different color combination possibilities. Even though the puzzle has not changed size, the set up of the solution has changed the group that these solutions are isomorphic to. Once we change the set up to having a row, column, or diagonal translate in a different direction than the rest of the cubes, we are creating an entirely different group because now we cannot say L*F=B, meaning we have 27 elements instead of 9 because B is a new element. We can say B is a new element because it is not a color combination we have seen before. With this addition of elements to the group, these groups are now isomorphic to the group $Z_3 \oplus Z_3 \oplus Z_3$. The mapping for each element to its corresponding element with equal order is as follows:

$e \rightarrow (0,0,0)$	$L^2*F \rightarrow (1,2,0)$	$F^2*B^2 \rightarrow (2,0,2)$
$L \rightarrow (0,1,0)$	$L^2*F^2 \rightarrow (2,2,0)$	$L*F*B \rightarrow (1,1,1)$
$L^2 \rightarrow (0,2,0)$	$L*B \rightarrow (0,1,1)$	$L*F^2*B \rightarrow (2,1,1)$
$F \rightarrow (1,0,0)$	$L*B^2 \rightarrow (0,1,2)$	$L^2*F*B \rightarrow (1,2,1)$
$F^2 \rightarrow (2,0,0)$	$L^2*B \rightarrow (0,2,1)$	$L^2*F^2*B \rightarrow (2,2,1)$
$B \rightarrow (0,0,1)$	$L^2*B^2 \rightarrow (0,2,2)$	$L*F*B^2 \rightarrow (1,1,2)$
$B^2 \rightarrow (0,0,2)$	$F*B \rightarrow (1,0,1)$	$L*F^2*B^2 \rightarrow (2,1,2)$
$L*F \rightarrow (1,1,0)$	$F*B^2 \rightarrow (1,0,2)$	$L^2*F*B^2 \rightarrow (1,2,2)$
$L*F^2 \rightarrow (2,1,0)$	$F^2*B \rightarrow (2,0,1)$	$L^2*F^2*B^2 \rightarrow (2,2,2)$

It can be seen that by adding a new element B, we get three times as many elements as we had before, sort of adding a third dimension to the elements.

DISCUSSION

Overall, the conclusions of connecting this puzzle to group theory may seem simple, but it took a lot of deep investigating about the different ways to set up a solution for each puzzle. While we weren't able to get a confident number on all the possible ways to set up all the solutions to a 4x4x4 puzzle, we were able to say that we explored all possibilities for the 3x3x3 solutions, and therefore find that depending on which of the 28 set up of the solutions were chosen, it would be isomorphic to either $Z_3 \oplus Z_3$ or $Z_3 \oplus Z_3 \oplus Z_3$.

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